PERGAMON

Intl. Trans. in Op. Res. 6 (1999) 495–524

INTERNATIONAL TRANSACTIONS IN OPERATIONAL RESEARCH

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Markovian analysis of production lines with Coxian-2 service times

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Received 13 July 1998; received in revised form 14 January 1999; accepted 3 March 1999

Abstract

This paper is concerned with the analysis of reliable production lines. The service times at each station of the line are assumed to follow the Coxian-2 distribution. Raw material arrives at the first station of the line which is assumed that is never empty. Buffers of non-identical capacities are allowed between successive stations. The structure of the transition matrices of these specific type of production lines is examined and a recursive algorithm is developed for generating them, for any number of stations K. This method allows one to obtain the exact solution of a sparse linear system by the use of the Gauss–Seidel method. From the solution of these systems the throughput rate of the production lines is calculated. However, this algorithm is not computationally efficient as it is restricted by the size of the problem. The main contribution of this paper is the study of the transition matrices of production lines with Coxian service times. © 1999 IFORS. Published by Elsevier Science Ltd. All rights reserved.

Keywords: Production lines; Stochastic modeling; Performance evaluation; Markov chains; Quassi-birth-and-death (QBD) processes; Coxian distribution

1. Introduction and literature review

Numerous research articles have been devoted to the analysis and design of asynchronous production lines. Exact analytical results are available only for short lines (Avi-Intzhak and Yadin, 1965; Hunt, 1956; Konheim and Reiser, 1976; Latouche and Neuts, 1980; Muth, 1984

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PII: S0969-6016(99)00012-X

among others). The lack of exact results is mainly due to the blocking phenomenon which occurs due to the finite capacities of the intermediate buffers and which contributes greatly to the intractability of the mathematics involved.

Results for longer production lines are available for restrictive assumptions. Hillier and Boling (1967) provided a numerical approach for solving production lines with exponential and Erlang service times. Papadopoulos et al. (1989, 1990) and Heavey et al. (1993) used a similar approach. Other important references include the works by Altiok and Ranjan (1988), Gun and Makowski (1988), Altiok (1982), (1989), Brandwajn and Jow (1985, 1988), Gun and Makowski (1989), Onvural and Perros (1990) and Perros and Altiok (1986). All these works are described in detail and in a methodological manner in the book by Perros (1994).

This paper extends the algorithm of Papadopoulos et al. (1989) to include Coxian distributed processing lines. This in conjunction with the algorithms given by Papadopoulos et al. (1989, 1990) allow greater flexibility in the modelling of production lines. This is because the Coxian, the Erlang and the exponential distributions can be fitted to service times data where the coefficient of variation is greater, less or equal to 1, respectively.

The main contribution of this paper is the study of the transition matrices of production lines with Coxian service times and the development of an algorithm that calculates the throughput rate of these type of systems. However, this algorithm is not computationally efficient as it is restricted by the size of the problem. Although, we succeeded to generate the transition matrix of any K-station line, the algorithm for solving the resulting system of linear equations gives numerical results only for a limited number of stations.

The reason we used the Coxian-2 distribution has been its suitability for modelling possible machine breakdowns. Coxian-2 distribution belongs to the class of phase-type distributions. The first phase describes the service period whereas the second phase describes the repair period, which takes place whenever a station breakdown occurs with some probability d_2. Altiok and Stidham (1983) justified this suitability of the Coxian-2 distribution for modelling the total service time at workstations that are subject to breakdowns.

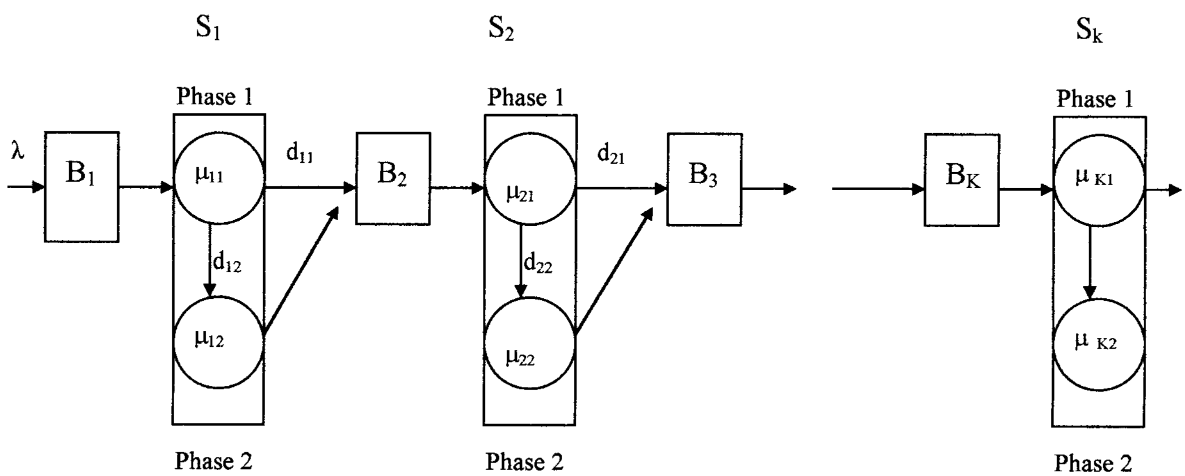

Fig. 1. A production line with K workstations, K buffers and Coxian-2 service times at each workstation.

The paper is organised as follows. Section 2 describes the model whereas Section 3 gives the number of states which corresponds to the order of the transition matrix of the Markovian process that describes the system under study. Section 4 describes the structure of the transition matrices and Section 5 presents the recursive algorithm for generating the transition matrices. Section 6 gives some numerical results for the throughput of short Coxian production lines and Section 7 concludes the paper and discusses directions for future research work. Finally, Appendix A gives an application of the proposed algorithm for generating the transition matrices of a three-station Coxian production line.

2. The model

The system to be modelled is a production line consisting of K workstations S_i, $i = 1,2,\ldots,K$, arranged in series with $K-1$ finite intermediate buffers $B_2,\ldots,B_K$ and one buffer, B_1, of infinite capacity in front of the first workstation (see Fig. 1).

Jobs arrive at the first station according to a homogeneous Poisson distribution with mean rate λ. Each item enters the line at station 1, passes through all stations in order and leaves the last station in finished form. The service times are assumed to be independent Coxian-2 distributed with mean values $1/\mu_{ij}$, $i = 1,2,\ldots,K, j = 1,2$ and branching probabilities d_{ij} where d_{i1} denotes the probability of a job leaving station i without entering the second phase of the Coxian-2 distribution, whereas d_{i2} is the probability of passing from the first to the second phase of the Coxian-2 distribution. (It holds: $d_{i2} = 1 - d_{i1}$.)

Although primarily the stations of the line are assumed perfectly reliable, the use of the Coxian-2 distribution allows the modelling of machine breakdowns. Jobs in all workstations are processed in a FIFO (first in-first out) manner and the well-known manufacturing type of blocking (or blocking after service, BAS) is assumed in every station. That is, a customer upon service completion at queue I attempts to join destination queue $\mathrm{I}+1$. Blocking will occur if at that instant queue $\mathrm{I}+1$ is full. The blocked customer is forced to wait at queue I until it enters at queue $\mathrm{I}+1$. During this time the ith server is blocked and it cannot serve any other customers that might be waiting in its queue.

2.1. Quasi-birth-and-death process

For modelling purposes, the system under study is split into two units: Unit I consisting of buffer B_1 and workstation S_1 and Unit II which consists of the remaining buffers and stations of the line, sometimes known as the subsystem.

The state of the system under consideration is given by a two-dimensional stochastic process $N(t) = \{N_1(t), N_2(t)\}$. Both co-ordinate random variables are integer valued and nonnegative. $N_1(t)$ represents the number of jobs in the first queue (Unit I) at time t and N_1 is the number of jobs in the first queue at equilibrium. There is no upper limit for N_1. N_2 represents the state of Unit II in equilibrium when N_1 is assumed constant. N_2 may assume only values from 1 to m where m is a finite positive integer, which denotes the number of customers in the sub-network (Unit II):

$$m = (K-1) + \sum_{i=2}^{K} B_i.$$

The changes in the state of the system are caused by the occurrence of various events. The occurrence times for all events have negative exponential distributions with strictly positive means. Thus, the process is Markovian. Its state-space is

$$S = \left\{(i,j) = i \geq 0, 1 < j \leq m\right\},$$

with the index i specifying the total number of items queued up or in service in the first workstation (Unit I). Such items are called 'I customers'. The index j specifies the state of Unit II. It is important to note that upon completion of service in Unit I, an 'I customer' becomes a 'II customer'.

The transition matrix that describes the system has the following block tri-diagonal form (see Evans, 1967; Neuts, 1981; Latouche and Neuts, 1980):

$$P = \begin{bmatrix} A_{01} & A_0 & & & & & \\ A_2 & A_1 & A_0 & & & & \\ & A_2 & A_1 & A_0 & & & \\ & & A_2 & A_1 & A_0 & & \\ & & & & \cdots & \cdots & \cdots \end{bmatrix} \tag{1}$$

The equilibrium equations $\boldsymbol{\pi P} = \mathbf{0}$ can be expressed in matrix-difference form as

$$\boldsymbol{\pi}_{\mathrm{I}} A_0 + \boldsymbol{\pi}_{\mathrm{I}+1} A_1 + \boldsymbol{\pi}_{\mathrm{I}+2} A_2 = 0, \quad \text{for } \mathrm{I} = 0,1,2,\ldots \tag{2}$$

and

$$\boldsymbol{\pi}_0 A_{01} + \boldsymbol{\pi}_1 A_2 = 0 \tag{3}$$

for the boundary equations.

A_0 is a square matrix describing the transitions in the sub-network which simultaneously produce inputs to the first queue. A_1 is a square matrix describing transitions in the sub-network which neither produce inputs nor outputs from the first queue assuming that the queue is not empty. A_2 is a square matrix describing transitions in the sub-network which simultaneously produce outputs from the first queue. A_{01} is a square matrix describing transitions in the sub-network which produce neither inputs nor outputs from the first queue, assuming that the queue is *empty*.

A Markov chain whose equilibrium equations have the form of Eqs. (2) and (3) was termed by Wallace (1973) a Quasi-Birth-and-Death (QBD) process.

Evans (1967) proposed that a geometric solution of the form

$$\boldsymbol{\pi}_{\boldsymbol{i}} = \boldsymbol{\pi}_{\mathbf{0}} R^i, \quad i \geq 0 \tag{4}$$

exists under certain conditions and he offered an algorithm based on finding a root-matrix, R^*, which is the minimal nonnegative solution to the matrix quadratic equation

Table 1
Notation

Symbol	Meaning
K	Number of stations
S_i	Workstation i
B_i	Buffer capacity preceding the ith station
s_i	Status of station i
c_i	Number of items queued up or in service at workstation S_i including the blocked item at station S_{i-1} (if any)
$M^K_{B_2,B_3,\ldots,B_K}$	Number of states in the sub-network of a K-station system with buffer capacities $B_2,\ldots,B_K$ of the intermediate buffers

$$R^2A_2 + RA_1 + A_0 = 0 \tag{5}$$

2.2. Calculation of the throughput

Solving the system

$$\boldsymbol{\pi}A = 0, \qquad \boldsymbol{\pi}\boldsymbol{e} = 1 \tag{6}$$

where A is the conservative stable matrix given by

$$A = A_0 + A_1 + A_2 \tag{7}$$

and $\boldsymbol{e}$ is a $m \times 1$ column vector with all elements equal to 1, will give explicit results for $\boldsymbol{\pi}$.

The equilibrium condition is given by

$$\boldsymbol{\pi}A_2\boldsymbol{e} > \boldsymbol{\pi}A_0\boldsymbol{e} \tag{8}$$

From this relationship the critical input rate to the system can be determined. In the steady-state, this critical input rate is identical to the maximum throughput rate of the saturated production line (i.e., its first station is never empty).

2.3. Notation

The notation used in the rest of this article is listed in Table 1.

The following $1 \times 2K$ vector describes the states of the system:

$$(c_1,s_1,c_2,s_2,\ldots,c_K,s_K) \tag{9}$$

$s_i,i \neq 1$ can take any value from 0 to 2.Table 2 lists the value $s_i,i \neq 1$ can take and an explanation of their meaning.

As it is assumed that station 1 is never idle, s_1 will never equal 0.

$c_1 = 1,2,\ldots,\infty$ and $c_i = 0,1,\ldots,B_i,B_i+1,\ B_i+2$ for $i = 2,3,\ldots,K$; $c_i = B_i+1$ means that buffer B_i is full and station S_i is busy, whereas $c_i = B_i+2$ has the same meaning as above plus that station S_i is blocked.

Table 2
Possible states of station i

S_i	Meaning
0	Station i is idle
1	Station i is busy and in phase 1
2	Station i is busy and in phase 2

3. Number of states

This section gives an expression that calculates the number of states of the system under consideration. We introduce a schematic representation of the states of the system. More specifically, the 'states tree' represents the states of the system. Fig. 2 shows the 'states tree' for two-station line with no intermediate buffers. We examine this simple system for the number of states initially and show that expressions derived for the general case of $K > 2$ stations and inter station buffers with finite, non-zero and non-identical capacities give the correct number of states for this simple system.

In Fig. 2, the numbers in italics are the c_2 and s_2 and are concerned with the second workstation. $c_1 = 1, s_1 = 1$ or 2; $c_2 = 0,1,2, s_2 = 0,1,2$.

The number of states for a two-station line with no intermediate buffers is 10 and is given, in general, by

$$M_{B_2}^2 = 2\big[2(B_2 + 2) + 1\big] = 2(2B_2 + 5) \tag{10}$$

For $K = 3$ and $B_2 = B_3 = 0$, the first part of the 'states tree' of the sub-network is given in Fig. 3 (due to space considerations only one of the two similar sub-trees is shown).

From Fig. 3 one may observe that the first sub-tree of the three-station line consists of $(M_{B_2}^2/2) - 2 \times 1 = (B_2 + 2) + 1 = 3$ sub-sub-trees. The first of these three sub-sub-trees contains $(M_{B_3}^2/2) - 2 \times M^1 = (M_0^2/2) - 2 \times 1 = (10/2) - 2 = 3$ leaves (states). The other two sub-sub-

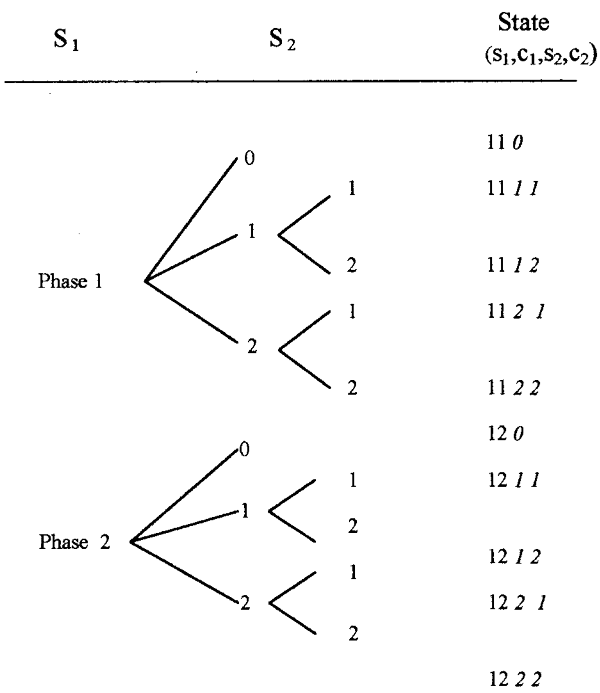

Fig. 2. The 'states tree' for $K = 2$ and $B_i = 0, i = 2, \ldots, K$.

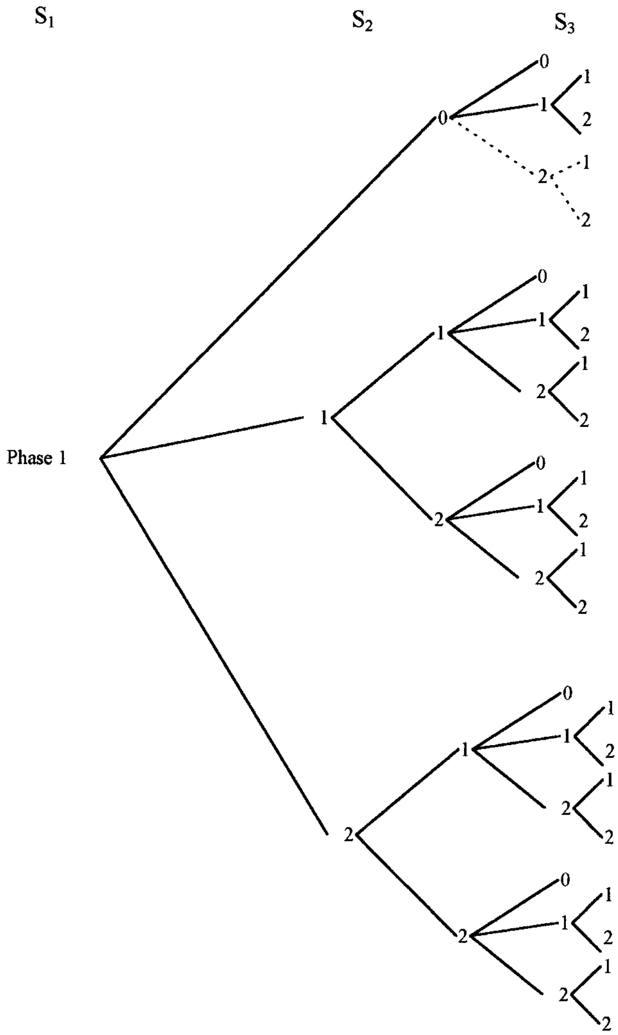

Fig. 3. The first part of the 'states tree' for a line with $K=3$ and $B_2 = B_3 = 0$.

trees have leaves, each. Thus, the first sub-tree contains $(M^2_{B_3}/2) - 2 + M^2_{B_3} + M^2_{B_3} = 23$ states. The complete 'states tree' contains the following number of states:

$$M^2_{B_2B_3} = 2\left(\frac{M^2_{B_3}}{2} - 2 + M^2_{B_3} + M^2_{B_3}\right) = 5 \times M^2_{B_3} - 4 = \frac{M^2_{B_3}}{2} M^2_{B_3} - 2 \times 2 \tag{11}$$

From this analysis the following recursive relationship (of second order) is derived:

$$M^K_{B_2B_3,\ldots,B_k} = \frac{M^2_{B_2}}{2} M^{K-1}_{B_3,\ldots,B_k} - 2 \times M^{K-2}_{B_4,\ldots,B_k} - 2, \quad K \geq 3 \tag{12}$$

with initial terms:

$$M^2_{B_2} = 2(2 \times B_2 + 5) \tag{13}$$

and

$$M^1 = 2. \tag{14}$$

Relationship (12) may be expressed as follows: the number of states of a system with K workstations consists of $M^2_{B_2}/2$ times the number of states of a system with $K-1$ workstations less twice the number of the states of a system with $K-2$ workstations.

It is easy to justify why we subtract the two sets of states in Eq. (12) in the $K=3$, $B_2 = B_3 = 0$ system. These correspond to non-feasible states and in Fig. 3 are depicted by dotted lines. It does not make sense for station S_2, to be blocked without having a customer at the workstation. Thus, 11021 and 11022 are non-feasible states of the three-station line with no intermediate buffers. For the same reason, from the second sub-tree, 12021 and 12022 are non-feasible states of the system.

Example Consider a production line with $K=4$ stations and intermediate buffers of capacities $B_2 = 1$,$B_3 = 0$ and $B_4 = 2$. From Eq. (12):

$$M^4_{1,0,2} = \frac{M^2_1}{2} M^3_{0,2} - 2 \times M^2_2 \tag{15}$$

$$M^3_{0,2} = \frac{M^2_0}{2} M^2_2 - 2 \times M^1 \tag{16}$$

From Eq. (13),

$$M^2_0 = 10, \quad M^2_1 = 14 \quad \text{and} \quad M^2_2 = 18$$

hence $M^3_{0,2} = 86$ as $M_1 = 2$

$$M^4_{1,0,2} = 566 \tag{17}$$

This example shows the recursive procedure that has to be applied in order to calculate the number of states for a system with K workstations and intermediate buffers of capacities $B_2, B_3, \ldots, B_K$, in terms of the respective number of states for systems with $K-1$ and $K-2$ stations and the same buffers (starting with the last buffer and working backwards).

This recursive scheme may be written as follows:

- $V_2 = 2(2 \times B_K + 5)$
- $V_3 = 2$
- DO $i = K, 3, -1$

- $V_1 = 2(2 \times B_i + 5)$
- $V = \frac{V_1 \times V_2}{2} - 2V_3$
- $V_3 = V_2$
- $V_2 = V$
- END DO
- $M^K_{B_2,B_3,\ldots,B_K} = V$

i.e., to calculate $M^K_{B_2,B_3,\ldots,B_K}$ one has to determine $M^2_{B_K}$ first, then $M^3_{B_{K-1},B_K}$,$M^4_{B_{K-2},B_{K-1},B_K}$, and so on, by using the buffers B_i with the order $i = K,K-1,K-2,\ldots,2$.

3.1. Ordering of states

The ordering of states affects the structure of the conservative matrix A. The objective is to find an ordering of the states that simplifies the structure of a matrix A as much as possible. This could facilitate the development of an efficient algorithm for the generation of matrix A.

4. Structure of matrix A

Matrix A is obtained from the sum of matrices A_0, A_1 and A_2. Matrices A_0 and A_2 have simple structures whereas matrix A_1 has a rather complicated structure. Matrix A_1 is examined first and then matrices A_2 and A_0.

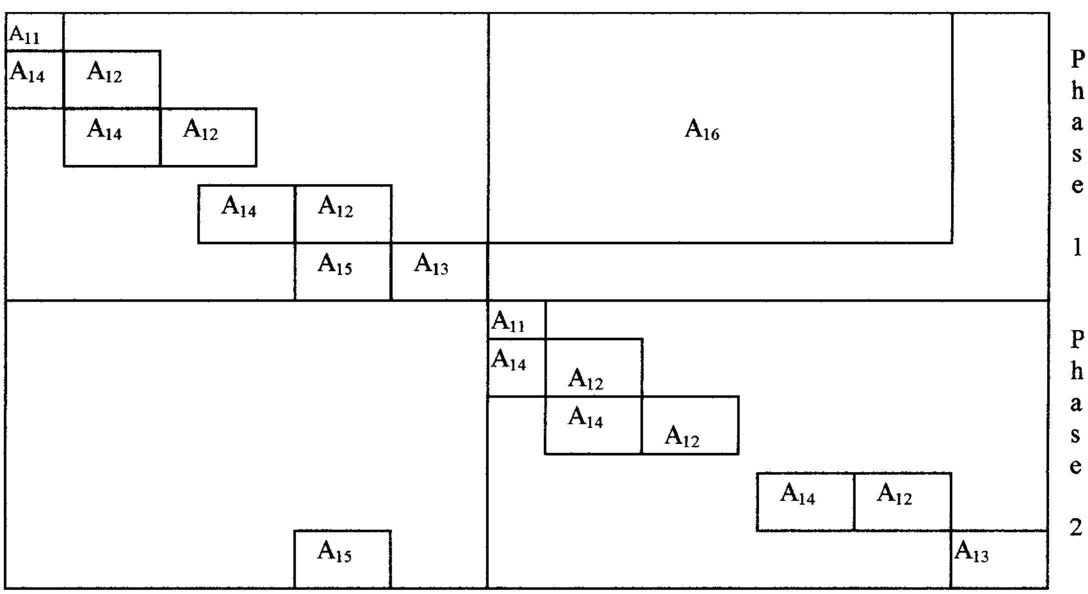

Fig. 4. Structure of matrix A_1 for $K > 2$.

4.1. Description of matrix A_1

Matrix A_1 for any K-station line ($K > 2$) with non-identical buffers takes the form described in Fig. 4. A_1 is a square matrix of order $M^K_{B_2,B_3,\ldots,B_K}$. It is split into two zones, which are almost the same. Thus is due to the two phases of the Coxian-2 distribution. The first zone of A_1 includes sub-matrices A_{11}, A_{12} (repeated B_2+1 times), A_{13}, A_{14} (repeated B_2+1 times) A_{15} and A_{16}. The second zone of A_1 contains the same sub-matrices of the first zone except A_{16} which does not appear. A_{15} is located at different cells to that in the first zone.

Dimension and location of sub-matrices A_{1i}, $i = 1,\ldots,6$.

- Sub-matrix A_{11} is an $m_{0,K} \times m_{0,K}$ matrix where $m_{0,K} = (M^{K-1}_{B_3,\ldots,B_K}/2) - M^{K-2}_{B_4,\ldots,B_K}$. A_{11} describes transitions within the sub-network when S_2 is empty and S_1 is busy. It consists of sub-matrices A_{11}, A_{12} and A_{14} for systems with $K-1$ stations (S_K,S_{K-1},$\ldots$,S_2) with the following changes: μ_{i1},μ_{i2},d_{i1} and d_{i2}, are replaced by $\mu_{i+1,1}$,$\mu_{i+1,2}$,$d_{i+1,1}$ and $d_{i+1,2}$, respectively, for $i = K,K-1,\ldots,2$. The generation of sub-matrix A_{11} is shown in Fig. 5. Note that $(A_{14})_{K-1}(A_{12})_{K-1}$ is repeated B_K+1 times.
- Sub-matrix A_{12} is a $m_{1,K} \times m_{1,K}$ matrix where $m_{1,K} = M^{K-1}_{B_K,B_{K-1},\ldots,B_3}$. A_{12} describes transitions in the sub-network or exits from the sub-network when S_2 and S_1 are busy. It consists of two zones. A_{12} is generated from sub-matrices A_{11},A_{12},A_{13},A_{14} and A_{16} for systems with $K-$ 1 stations (S_K,S_{K-1},$\ldots$,S_2) and sub-matrix F_{K-1} with the following changes:
  - ○ From all the diagonal elements of sub-matrix A_{12} of the first zone the element μ_{11} is subtracted whereas from all diagonal elements of sub-matrix A_{12} of the second zone μ_{12} is subtracted.
  - ○ μ_{i1},μ_{i2},d_{i1} and d_{i2}, are replaced by $\mu_{i+1,1}$,$\mu_{i+1,2}$,$d_{i+1,1}$ and $d_{i+1,2}$, respectively, for $i = K,K-1,\ldots,2$.
  - ○ Matrix F_{K-1} of the first zone is a square diagonal matrix of order with its diagonal elements equal to $d_{(K-2),1} \cdot \mu_{(K-2),1}$. Matrix F_{K-1} of the second zone is a square diagonal matrix of order $M^{K-2}_{B_K,B_{K-1},\ldots,B_4}$ with its diagonal elements equal to $\mu_{(K-2)2}$.The generation of

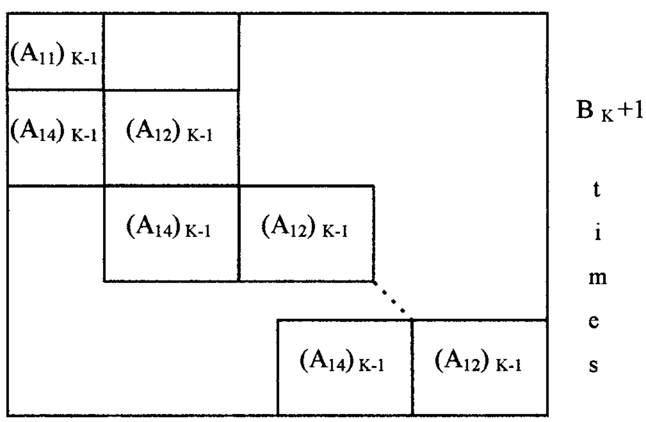

Fig. 5. Generation of $(A_{11})_K$ from sub-matrices $(A_{11})_{K-1}$, $(A_{12})_{K-1}$ and $(A_{14})_{K-1}$.

sub-matrix A_{12} is shown in Fig. 6.

- Sub-matrix A_{13} is an $m_{1,K} \times m_{1,K}$ matrix with $m_{1,K} = M^{K-1}_{B_K,B_{K-1},\ldots,B_3}$. A_{13} describes transitions within the sub-network or exits from the sub-network when S_2 is busy and S_1 is blocked. Due to the latter event, A_{13} is the same as A_{12} except that μ_{11} is subtracted from all the diagonal elements of A_{12} in the first zone and μ_{12} is subtracted from all the diagonal elements of A_{12} in the second zone.

  Sub-matrix A_{14} is located next to sub-matrix A_{12} (see Fig. 4) and is repeated $B_2 + 1$ times. A_{14} is a square matrix of order $m_{1,K} = M^{K-1}_{B_{K-1},B_{K-2},\ldots,B_2}$, with the only exception that the first sub-matrix A_{14} (from the $B_2 + 1$) is an $m_{0,K} \times m_{1,K}$ matrix. A_{14} describes transitions within or exits from the sub-network when station S_2 finishes its service and the serviced unit either enters the next buffer B_3 or, if this is empty, the next station S_3. A_{14} is split into two zones and is generated from A_{14} for systems with $K - 1$ stations. The structure of sub-matrix A_{14} is shown in Fig. 7.

  Sub-matrix A_{14} consists of (a) B_{K-1} diagonal matrices of dimensions $m_{1,K-1} \times m_{1,K-1}$ (except the first one, the dimension of which is $m_{0,K-1} \times m_{0,K-1}$) with all diagonal elements equal to $d_{(K-2),1} \cdot \mu_{(K-2),1}$ in the first zone and equal to $\mu_{(K-2),2}$ in the second zone and (b) from sub-matrix $(A_{14})_{K-1}$, i.e., with $K - 1$ workstations, in which the following changes are applied: $d_{i1} \cdot \mu_{i1}$ is replaced by $d_{i+1,1} \cdot \mu_{i+1,1}$.

- Sub-matrix A_{15} is located next to sub-matrix A_{13} (see Fig. 4). A_{15} is a square matrix of order $m_{1,K} = M^{K-1}_{B_K,B_{K-1},\ldots,B_3}$. It describes transitions within the sub-network or exits from the sub-network when station S_2 finishes its service and the serviced unit enters the subsequent buffer B_3. It is the same as sub-matrix A_{14}.

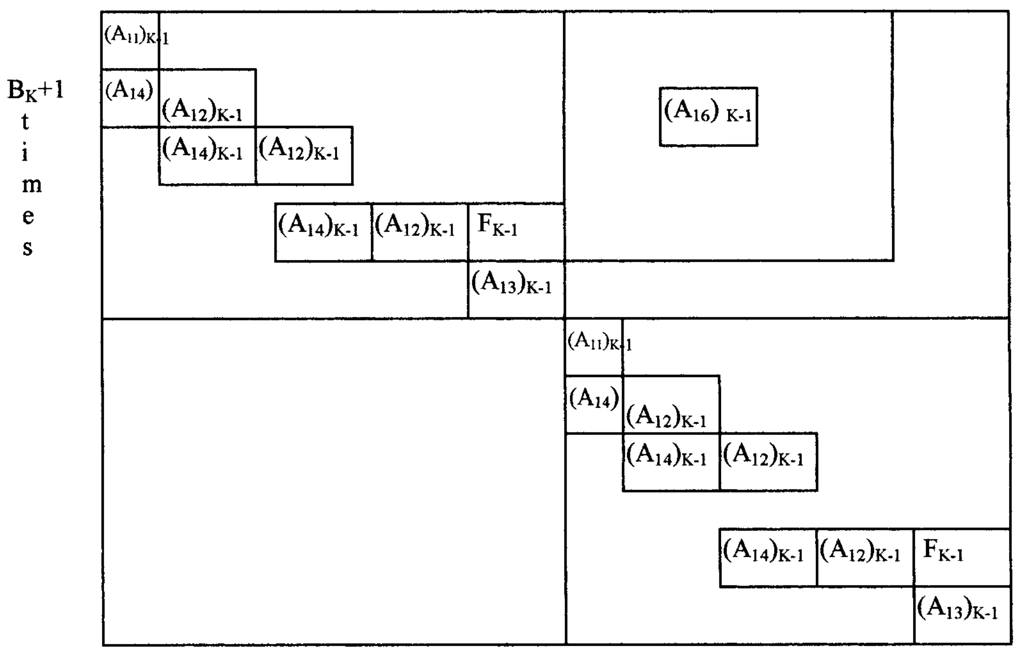

Fig. 6. Generation of $(A_{12})_K$ from sub-matrices $(A_{11})_{K-1}$, $(A_{12})_{K-1}$, $(A_{13})_{K-1}$, $(A_{14})_{K-1}$, $(A_{16})_{K-1}$ and F_{K-1}.

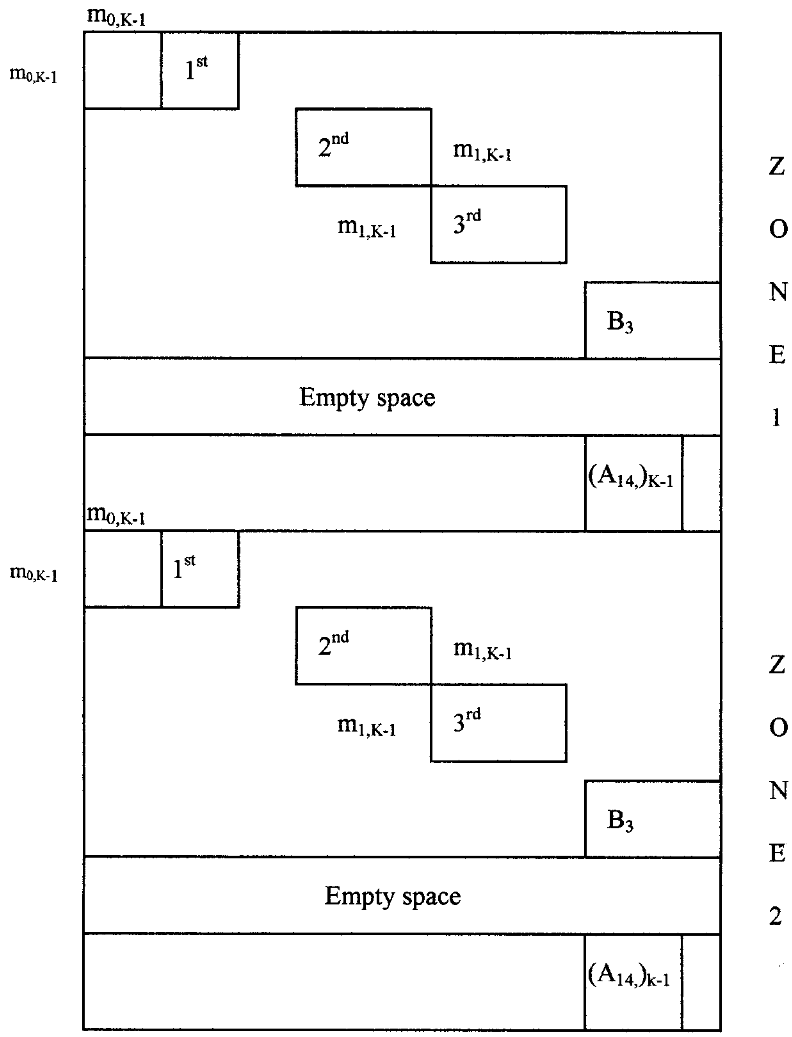

Fig. 7. The structure of sub-matrix A_{14}.

- Sub-matrix A_{16} is a square diagonal matrix of order $(M^{K}_{B_K,B_{K-1},\ldots,B_2}/2) - M^{K-1}_{B_{K-1},B_{K-2},\ldots,B_2}$ with all its diagonal elements equal to $d_{12} \cdot \mu_{11}$. It describes the transitions from phase 1 to phase 2 in Unit I. This event requires that the first station is not blocked.

4.2. Description of matrices A_0 and A_2

Matrix A_0 is a square diagonal matrix of order $M^{K}_{B_K,B_{K-1},\ldots,B_2}$ with all its diagonal elements equal to λ (= the mean arrival rate to Unit I). A_0 describes the arrivals of the customer to Unit I. Matrix A_2 is an orthogonal almost diagonal matrix with dimensions

$$\left(M^K_{B_2,\ldots,B_K} - M^{K-1}_{B_3,\ldots,B_K}\right)\frac{M^K_{B_2,\ldots,B_K}}{2}.$$

A_2 describes the flows of 'I customers' to Unit II, thereby becoming 'II customers'. It consists of two zones. The first zone describes the transitions of units from the first phase of service of the first station to the sub-network, whereas, the second zone describes the transitions of 'I customers' to Unit II from the second phase of service of the first station. Fig. 8 shows the structure of the A_2 matrix.

Matrix A_2, in its first zone, consists of $B_K + 2$ diagonal sub-matrices with their diagonal elements equal to $d_{11} \cdot \mu_{11}$. In the second zone the $B_K + 2$ diagonal sub-matrices have their elements equal to μ_{12}. The location of the non-zero elements of A_2 matrix is given by:

1. At the first zone: Place $d_{11} \cdot \mu_{11}$ at cells (i,j) with:

$$i = 1,2,\ldots,m_{0,K-1}, j = m_{0,K-1} + 1, m_{0,K-1} + 2,\ldots,2m_{0,K-1}$$

and

$$i = m_{0,K-1} + 1,\ldots,\frac{M^K_{B_K,B_{K-1},\ldots,B_2}}{2}, j = m_{0,K-1} + m_{1,K-1} + 1,\ldots$$

2. At the second zone: Place μ_{12} at cells (i,j) with:

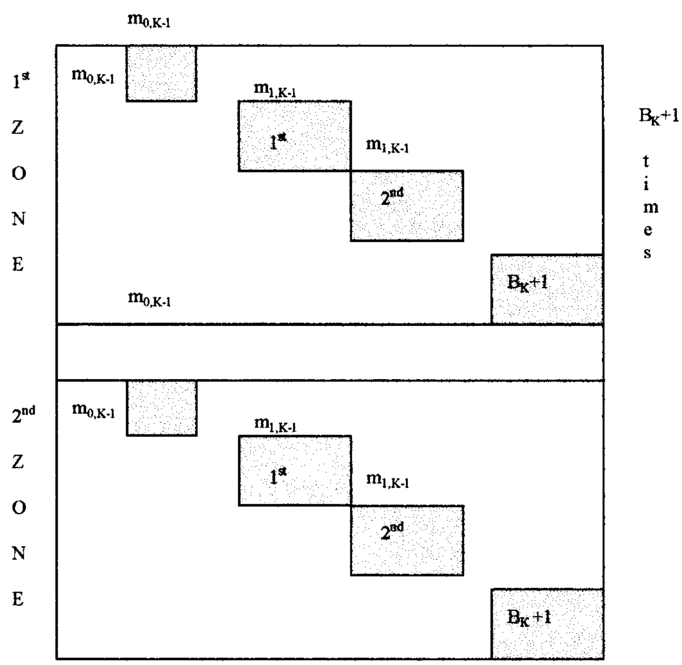

Fig. 8. The structure of matrix A_2.

$$i = \frac{M_{B_K,B_{K-1},\ldots,B_2}}{2} + 1, \ldots, + m_{0,K-1}, j = m_{0,K-1} + 1, m_{0,K-1} + 2, \ldots, 2m_{0,K-1}$$

and

$$i = \frac{M^K_{B_K,B_{K-1},\ldots,B_2}}{2} + m_{0,K-1} + 1, \ldots, i = +m_{0,K-1} + 1, \ldots j = m_{0,K-1} + m_{1,K-1} + 1, \ldots$$

Conservative matrix A is the sum of matrices A_0, A_1 and A_2 and has the structure shown in Fig. 9.

In Fig. 9, the shaded sub-matrices form matrix A_2.

5. The algorithm for the generation of matrix A

This section describes the generation of matrix A by the use of a recursive algorithm. The algorithm generates matrix A for production lines with the service times following the Coxian-2 distributions. The algorithm starts with the creation of matrix A_1 for the last two stations (S_{K-1} and S_K). It continues with the creation of matrix A_1 for the last three stations (S_K, S_{K-1}, S_{K-2}) and so on up to the generation of matrix A for all the stations of the line.

The flow chart of the generating algorithm is given in Fig. 10. The algorithm consists of seven rules. Each rule contributes to the construction of a part of matrix A. The algorithm is implemented in FORTRAN and the user inputs K, the number of stations, $B_2,\ldots,B_K$, the

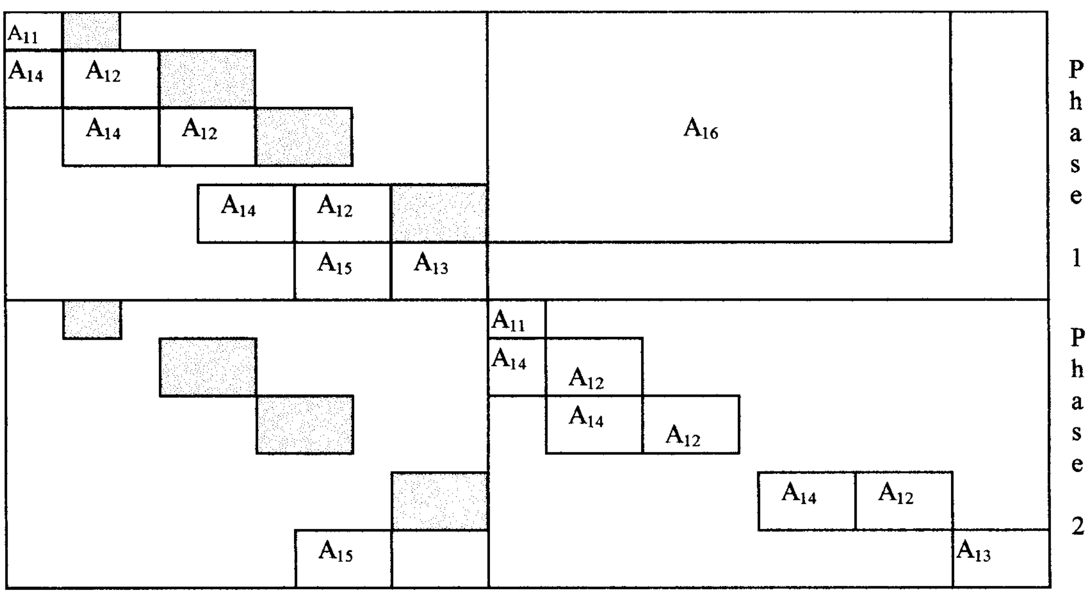

Fig. 9. Structure of transition matrix A.

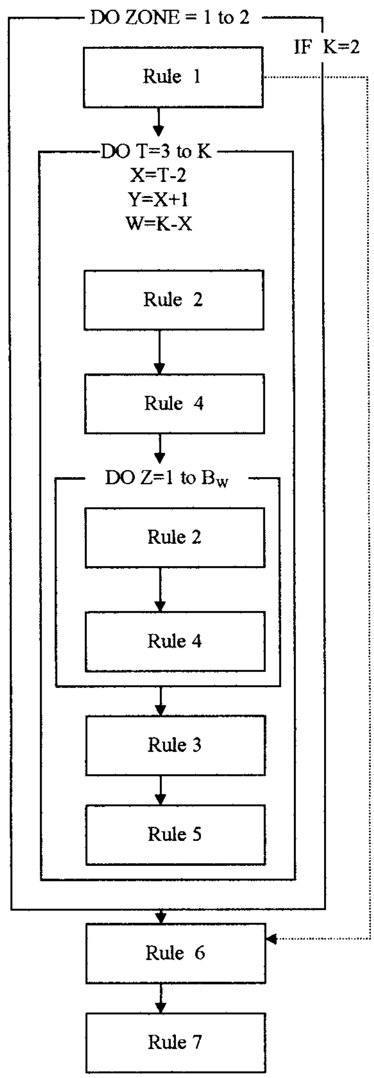

Fig. 10. Flow chart of the algorithm.

buffers capacities, μ_{ij}, the mean service rates of the two phases of service, d_{i1} and d_{i2}, the branching probability of the Coxian-2 distribution for $i = 1,2,\ldots,K$ and $j = 1,2$.

Rule 1 Generation of matrix A_1 for $K = 2$ stations or sub-matrix A_{11} for $K = 3$ stations.
Legend For convenience, (i,j) means element (i,j) of a matrix or a sub-matrix.

(1.1) The first element (row = 1, column = 1) is equal to $-\mu_{11}$.
(1.2) For the next two rows:
(1.2.1) For $i = 2,3$, respectively, put $(i,\, i)$ equal to $-\mu_{11} - \mu_{K1}$ and $-\mu_{11} - \mu_{K2}$.

(1.2.2) For $i = 2$, put $(i,i-1) = d_{K1} \cdot \mu_{K1}$ and $(i,i+1) = d_{K2} \cdot \mu_{K1}$.
(1.2.3) Put $(i,1) = \mu_{K2}$, for $i = 3$.
(1.3) For the next $2 \times B_K$ rows:
(1.3.1) For $i = 4,5$, put (i, i) equal to $-\mu_{11} - \mu_{K1}$ and $-\mu_{11} - \mu_{K2}$, respectively.
(1.3.2) For $i = 4$, put $(i,i-2) = d_{K1} \cdot \mu_{K1}$ and $(i,i+1) = d_{K2} \cdot \mu_{K1}$.
(1.3.3) Place $(i,i-3) = \mu_{K2}$, for $i = 5$.
The last three sub-rules (1.3.1)–(1.3.3) are applied B_K times. Each time, the sub-matrix generated is placed to the right diagonal and under the sub-matrix that has been previously generated.
(1.4) If $K = 2$ then
(1.4.2) Generation of sub-matrix A_{13} for $K = 2$. Rules (1.3.1)–(1.3.3) are repeated once more with the following changes: From the last two diagonal elements, μ_{K1} and μ_{K2} are subtracted.
(1.4.3) Generation of the second zone of matrix A_1 for $K = 2$. The first diagonal element of the second zone is cell (i, i) with $i = 2(B_K + 3)$. Rules (1.1)–(1.4.1) are applied again with the following changes: (a) i is replaced by $i + 2(B_K + 2) + 1$ and μ_{11} by μ_{12} and (b) applying Rule (1.4.1) (for generating the second zone of A_{13}), Rule (1.3.2) now becomes: for $i = 2(2B_K + 5) - 1$, place $(i,i-2(B+3)-1) = d_{K1}\mu_{K1}$ and $(i,i+1) = d_{K2} \cdot \mu_{K1}$, whereas, Rule (1.3.3) is replaced by: for $i = 2(2B_K + 5)$, place $(i,i-2(B_K+4)) = \mu_{K2}$. Then Rule 6 is applied to generate sub-matrix A_{16} as illustrated below.

Using Rules 1 and 6, matrix A_1 is generated for a production line with $K = 2$ stations. When $K > 2$, Rule 1 creates sub-matrix A_{11} for $K = 3$ stations. Rules 2–5 are then applied as illustrated in a DO-loop below.

DO $T = 3$ to K
$X = T - 2$
$Y = X + 1$
$W = K - X$
Rules 2–5
END DO

T and X are counters for the number of stations in the system and the sub-network, respectively. (For $X = 1$, the sub-network consists of stations S_K, S_{K-1}, for $X = 2$, S_K, S_{K-1}, S_{K-2} and so on, for $X = K-2$, $S_K, \ldots, S_2$.) Clearly $Y = X + 1$. W is an index giving the first station of the sub-network.

Rule 2 Generation of sub-matrix A_{12}.

(2.1) Sub-matrix A_{11} which was generated by Rule 1 is re-constructed with a single change. μ_{w1} is subtracted from all the diagonal elements of this square sub-matrix the order of which is

$$\left(\frac{M^{Y}_{B_{W+1},\ldots,B_K}}{2} - M^{Y-1}_{B_{W+2},\ldots,B_K}\right).$$

(2.2) The square diagonal matrix F_Y is created. F_Y is of order with all diagonal elements equal to $d_{w1} \cdot \mu_{w1}$. Its position with respect to $(A_{11})_Y$ is shown in Fig. 11.
(2.3) Sub-matrix $(A_{13})_Y$ is created which is a square matrix of order $M^{Y-1}_{B_{W+2},\ldots,B_K}$ (see Rule 1.2).
(2.4) Diagonal sub-matrix $(A_{16})_Y$ is generated, the order of which is $(M^{Y}_{B_{W+1},\ldots,B_K})/2 - M^{Y-1}_{B_{W+2},\ldots,B_K}$ and all its diagonal elements are equal to $d_{w2} \cdot \mu_{w1}$.

At this stage, the generation of the first zone of $(A_{12})_Y$ has been completed. To generate the second zone of $(A_{12})_Y$ steps 1–3 are repeated with the change that μ_{w1} is replaced by μ_{w2}. The relevant position of sub-matrices $(A_{11})_Y$, F_Y, $(A_{13})_Y$ and $(A_{16})_Y$ is shown in Fig. 11.

Rule 4 Generation of sub-matrix A_{14}.

Sub-matrix A_{14} is located next to A_{12} (see Fig. 4) and is repeated $B_2 + 1$ times. A_{14} is a square matrix of order $m_{1,T} \times m_{1,T} = M^{Y}_{B_{W+1},\ldots,B_K} \times M^{Y}_{B_{W+1},\ldots,B_K}$. Only the first sub-matrix A_{14} is an orthogonal matrix of dimensions

$$m_{1,T} \times m_{0,T} = M^{Y}_{B_{W+1},\ldots,B_K}\left(\frac{M^{Y}_{B_{W+1},\ldots,B_K}}{2} - M^{Y-1}_{B_{W+2},\ldots,B_K}\right).$$

(4.1) Generate a square diagonal matrix of order $m_{0,Y} = (\frac{M^{Y-1}_{B_{W+2},\ldots,B_K}}{2} - M^{Y-2}_{B_{W+3},\ldots,B_K})$ with all its diagonal elements equal to $d_{w1} \cdot \mu_{w1}$. Place the top-left elements of this matrix at cell:

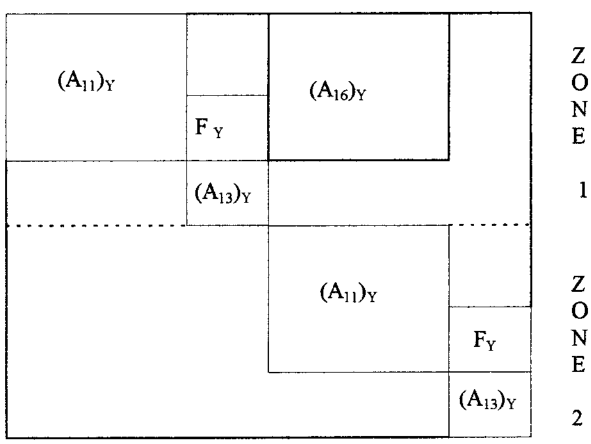

Fig. 11. Sub-matrices generated by Rule 2.

$$(m_{0,Y}+1,2)=\left(\frac{M^{Y-1}_{B_{W+2},\ldots,B_K}}{2}-M^{Y-2}_{B_{W+3},\ldots,B_K}+1,2\right).$$

(4.2) If $B_{W+1}\neq 0$, generate B_{W+1} diagonal matrices of order $m_{1,Y}=M^{Y-1}_{B_{W+2},\ldots,B_K}$ with all their diagonal elements equal to $d_{w1}\cdot\mu_{w1}$. Locate the top-left element of the first matrix at cell: $(m_{0,T}+m_{0,Y}+1, m_{0,Y}+m_{1,Y}+1)$.
(4.3) Generate a (2×1) matrix with elements $d_{K1}\cdot\mu_{K1}$ and μ_{K2}. Locate these elements at cells:

$$2\left(\frac{M^{Y}_{B_{W+1},\ldots,B_K}}{2}-M^{Y-1}_{B_{W+2},\ldots,B_K}\right)+1,\frac{M^{Y}_{B_{W+1},\ldots,B_K}}{2}-M^{Y-1}_{B_{W+2},\ldots,B_K}-1$$

and

$$2\left(\frac{M^{Y}_{B_{W+1},\ldots,B_K}}{2}-M^{Y-1}_{B_{W+2},\ldots,B_K}\right)+2,\frac{M^{Y}_{B_{W+1},\ldots,B_K}}{2}-M^{Y-1}_{B_{W+2},\ldots,B_K}-1.$$

At this stage, the first zone of (A_{14}) for T stations has been generated. To generate the second zone of A_{14}, the first zone is copied to the next $M^{Y}_{B_{W+1},\ldots,B_K}/2$ rows with the change that $d_{w1}\cdot\mu_{w1}$ is replaced by μ_{w2}. This concludes the generation of the second zone of A_{14} and of $(A_{14})_T$ itself.

Rules 2 and 4 are included in a DO-loop:

DO $Z=1$ to B_W
Rules 2 and 4: Copy sub-matrices A_{12} and A_{14}, B_W times.
END DO

Note: If $B_W=0$, Rules 2 and 4 are not applied.

Rule 3 Generation of sub-matrix A_{13}.

Sub-matrix A_{13} describes transitions within the sub-network or exits from the sub-network when station S_1 is blocked. A_{13} is similar to A_{12}; the only difference is that μ_{11} is subtracted from all the diagonal elements of A_{12}. A_{13} is located diagonally to the right and under matrix A_{12} (see Fig. 4).

Rule 5 Generation of sub-matrix A_{15}.

Sub-matrix A_{15} is exactly the same as A_{14} with respect to its order and its elements. The only difference is its location, which is shown in Fig. 4.

Rule 6 Generation of sub-matrix A_{16}.

Sub-matrix A_{16} is a diagonal matrix of order $(M^K_{B_2,\ldots,B_K}/2) - M^{K-1}_{B_3,\ldots,B_K}$ with all its diagonal elements equal to $d_{12} \cdot \mu_{11}$. The location of A_{16} is shown in Fig. 4.

At this stage, the first zone of matrix A_1 has been completed, for K stations. To generate the second zone of matrix A_1, Rules 1, 2, 4, 2 and 4, 3 and 5 are applied with the following changes:

(a) μ_{11} is replaced by μ_{12} and
(b) the location of sub-matrix A_{15} is changed. This is located now under A_{15} of the first zone (see Fig. 4).

Rule 7 Generation of matrix A_2.

(i) For the first zone: Place $d_{11} \cdot \mu_{11}$ at cells (i, j) with:

$$i = 1,2,\ldots,m_{0,K-1}, j = m_{0,K-1} + 1, m_{0,K-1} + 2,\ldots,2m_{0,K-1}$$

and

$$i = m_{0,K-1} + 1,\ldots,\frac{M^K_{B_K,B_{K-1},\ldots,B_2}}{2} - M^{K-1}_{B_{K-1},B_{K-2},\ldots,B_2},$$

$$j = m_{0,K-1} + m_{1,K-1} + 1,\ldots,\frac{M^K_{B_K,B_{K-1},\ldots,B_2}}{2}.$$

(ii) For the second zone: Place μ_{12} at cells (i, j) with:

$$i = \frac{M^K_{B_K,B_{K-1},\ldots,B_2}}{2} + 1,\ldots,\frac{M^K_{B_K,B_{K-1},\ldots,B_2}}{2} + m_{0,K-1},$$

$$j = m_{0,K-1} + 1, m_{0,K-1} + 2,\ldots,2m_{0,K-1}$$

and

$$i = \frac{M^K_{B_K,B_{K-1},\ldots,B_2}}{2} + m_{0,K-1} + 1,\ldots,M^K_{B_K,B_{K-1},\ldots,B_2} - M^{K-1}_{B_K,B_{K-1},\ldots,B_3},$$

$$j = m_{0,K-1} + m_{1,K-1} + 1,\ldots,\frac{M^K_{B_K,B_{K-1},\ldots,B_2}}{2}.$$

This completes the description of the Algorithm.

6. Numerical results

In this section numerical results are presented concerning the throughput of production lines with $K = 3,4,5$ and 6 stations and different values of intermediate buffers. In all systems considered the rate of the first phase (i.e., the service rate) is assumed equal to one ($\mu_{i1} = 1$), whereas the rate of the second phase (i.e., the repair rate) takes the values $\mu_{i2} = 0.04$, 0.1, 0.2, 1, for $i = 1,2,\dots,K$. Actually, the time of the second phase (mean repair time) is $t_2 = 1/\mu_{i2} = 25$, 10, 5, 1 times larger than the mean (service) time of the first phase. The transition probability from phase 1 to phase 2 (i.e., the probability that the service is interrupted by a failure) is assumed — for computational convenience — identical for all the stations and takes the values $d_{i2} = 0.004$, 0.01, 0.05 $\kappa\alpha\iota$ 0.10.

The number of states of the resulting Markov processes grows rapidly with the number of stations and the buffer capacities, which makes the system very difficult and sometimes impossible to analyze. This may be due to the ordering of the states of the underlying Markovian process that describes the system which leads to a convergence problem. We strongly believe that if we overcome this obstacle, we will be able to solve larger systems than Coxian lines with six stations in series and only one buffer slot allocated in one of the five intermediate buffers. We justify this from the very short run times taken to generate and solve these lines (with six stations and one buffer slot). The maximum run time has been 1.22 s on a 166 MHz Pentium MMX computer.

Table 3
The throughput of a line with $K = 3$ stations, $\mu_{11} = \mu_{21} = \mu_{31} = 1$, $\mu_{12} = \mu_{22} = \mu_{32} = 0.04$, 0.1, 0.2, 1, $d_{11} = d_{21} = d_{31} = 0.996$ and $d_{12} = d_{22} = d_{32} = 0.004$

B_2	$1/\mu_2$	B_3			
		0	1	2	3
		X_3	X_3	X_3	X_3
0	25	0.48516	0.52198	0.54017	0.55026
	10	0.53125	0.57551	0.59743	0.60947
	5	0.54838	0.59548	0.61873	0.63132
	1	0.56186	0.61089	0.63483	0.64752
1	25	0.52198	0.56358	0.58534	0.59836
	10	0.57551	0.62631	0.65305	0.66899
	5	0.59548	0.64987	0.667848	0.69541
	1	0.61089	0.66780	0.69753	0.71491
2	25	0.54017	0.58534	0.60945	0.62421
	10	0.59743	0.65305	0.68297	0.70127
	5	0.61873	0.667848	0.71063	0.73021
	1	0.63483	0.69753	0.73110	0.75134
3	25	0.55026	0.59836	0.62421	0.64020
	10	0.60947	0.66899	0.70127	0.72125
	5	0.63132	0.69541	0.73021	0.75166
	1	0.64752	0.71491	0.75134	0.77362

7. Conclusions and further research

In this article, the algorithm of Papadopoulos et al. (1989, 1990) has been extended to include Coxian-2 distributed service times. The structure of the conservative matrix A has been examined and a recursive algorithm has been developed for its generation. This allows the analyst to obtain results for a wide range of models — with the algorithm being very general allowing all parameters of the system to vary arbitrarily. This algorithm supplements the algorithm by Papadopoulos et al. (1989, 1990) in that it allows one to model service times distributions with coefficients of variation greater than unity.

The main contribution of this paper is the development of the recursive algorithm that generates the transition matrix of any K-station production line with any number of buffer positions between any two successive stations of the line. However, the algorithm that calculates the throughput rate of production lines with Coxian-2 service times is not

Table 4
The throughput of a line with, $K = 4$ stations, $\mu_{11} = \mu_{21} = \mu_{31} = \mu_{41} = 1$, $\mu_{12} = \mu_{22} = \mu_{32} = \mu_{42} = 0.04$, 0.1, 0.2, 1, $d_{11} = d_{21} = d_{31} = d_{41} = 0.996$ and $d_{12} = d_{22} = d_{32} = d_{42} = 0.004$

B_2	B_3	$1/\mu_2$	B_4			
			0	1	2	3
			X_4	X_4	X_4	X_4
		25	0.43094	0.45239	0.46116	0.46537
0	0	10	0.48001	0.50655	0.51724	0.52212
		5	0.49853	0.52701	0.53828	0.54319
		1	0.51273	0.54242	9.55383	0.55848
		25	0.46235	0.48404	0.49342	0.49826
0	1	10	0.51903	0.54627	0.55789	0.56366
		5	0.54051	0.56989	0.58222	0.58811
		1	0.55664	0.58736	0.59992	0.60563
		25	0.47979	0.50105	0.51017	0.51499
0	2	10	0.54078	0.56766	0.57898	0.58473
		5	0.56384	0.59286	0.60486	0.61071
		1	0.58084	0.61116	0.62332	0.62899
		25	0.45239	0.47904	0.49077	0.49671
1	0	10	0.50655	0.54001	0.55468	0.56189
		5	0.52701	0.56318	0.57889	0.58639
		1	0.54242	0.58042	0.59662	0.60405
		25	0.48404	0.51216	0.52515	0.53219
1	1	10	0.54627	0.58219	0.59874	0.60752
		5	0.56989	0.60897	0.62684	0.63613
		1	0.58736	0.62864	0.64722	0.65661
		25	0.50105	0.53102	0.54351	0.55088
1	2	10	0.56766	0.60514	0.62234	0.63163
		5	0.59286	0.63382	0.65245	0.66232
		1	0.61116	0.65456	0.67398	0.68401

computationally efficient as it is restricted by the size of the line. The reason we used the Coxian-2 distribution has been its suitability for modelling machine breakdowns. The first phase of the distribution can model the service period whereas the second phase can model the repair period, which takes place whenever the machine breaks down. The latter occurs with probability d_{i2}, $i = 1,2,\ldots,K$ (see Tables 3–6).

An interesting area for further investigation would be to extend these three algorithms (by Papadopoulos et al. (1989, 1990) and the present one) to include workstations with multiple parallel servers and machines that are subject to breakdowns. Another area for future research would be to examine the buffer allocation problem in production lines with multiple servers at each workstation with the objection of meeting some appropriate decision criterion.

Table 5
The throughput of a line with, $K = 5$ stations, $\mu_{11} = \mu_{21} = \mu_{31} = \mu_{41} = \mu_{51} = 1$, $\mu_{12} = \mu_{22} = \mu_{32} = \mu_{42} = \mu_{52} = 0.04$, 0.1, 0.2, 1, $d_{11} = d_{21} = d_{31} = d_{41} = d_{51} = 0.95$ and $d_{12} = d_{22} = d_{32} = d_{42} = d_{52} = 0.05$

B_2	B_3	B_4	$1/\mu_2$	B_5			
				0	1	2	3
				X_5	X_5	X_5	X_5
			25	0.13437	0.13785	0.14027	0.14223
0	0	0	10	0.24715	0.25542	0.26015	0.26348
			5	0.34011	0.35313	0.35296	0.36280
			1	0.46314	0.48180	0.48771	0.48968
			25	0.13979	0.14303	0.14533	0.14722
0	0	1	10	0.26029	0.26785	0.27231	0.27551
			5	0.36130	0.37298	0.37866	0.38204
			1	0.49491	0.51086	0.51067	0.51796
			25	0.13979	0.14349	0.14602	0.14806
0	1	0	10	0.26029	0.26957	0.27485	0.27855
			5	0.36130	0.37670	0.38411	0.38845
			1	0.49491	0.51894	0.52743	0.53065
			25	0.13785	0.14159	0.14414	0.14620
1	0	0	10	0.25542	0.26475	0.27005	0.27375
			5	0.35313	0.36856	0.37596	0.38026
			1	0.48180	0.50575	0.51412	0.51722
			25	0.14303	0.14699	0.14967	0.15181
1	1	0	10	0.26785	0.27831	0.28424	0.28836
			5	0.37298	0.39121	0.40017	0.40546
			1	0.51086	0.54152	0.55340	0.55836
			25	0.14503	0.14850	0.15093	0.15289
0	1	1	10	0.27314	0.28178	0.28681	0.29038
			5	0.38236	0.39659	0.40357	0.40776
			1	0.52730	0.54894	0.55663	0.55976
			25	0.14850	0.15225		
1	1	1	10	0.28178	0.29175		
			5	0.39659	0.41421		
			1	0.54894	0.57929		

Table 6
The throughput of a line with, $K = 6$ stations, $\mu_{11} = \mu_{21} = \mu_{31} = \mu_{41} = \mu_{51} = \mu_{61} = 1$, $\mu_{12} = \mu_{22} = \mu_{32} = \mu_{42} = \mu_{52} = \mu_{62} = 0.04$, 0.1, 0.2, 1, $d_{11} = d_{21} = d_{31} = d_{41} = d_{51} = d_{61} = 0.99$ and $d_{12} = d_{22} = d_{32} = d_{42} = d_{52} = d_{62} = 0.01$

B_2	B_3	B_4	B_5	B_6	$1/\mu_2$	X_6
					25	0.28613
0	0	0	0	0	10	0.38021
					5	0.42518
					1	0.46211
					25	0.29241
0	0	0	0	1	10	0.39057
					5	0.43757
					1	0.47573
					25	0.29701
0	0	0	1	0	10	0.39846
					5	0.44727
					1	0.48680
					25	0.29870
0	0	1	0	0	10	0.40142
					5	0.45097
					1	0.49114
					25	0.29701
0	1	0	0	0	10	0.39846
					5	0.44727
					1	0.48680

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments which helped improve the appearance of the paper.

Appendix A. The application of the algorithm to a three-station production line

Here, the explicit derivation of the matrices and sub-matrices is presented for a three-station production line ($K = 3$) with intermediate buffers of capacities, $B_2 = B_3 = 1$. Service times follow the Coxian-2 distribution with mean service rates μ_{i1} and μ_{i2}, $i = 1, 2, 3$ with branching probabilities d_{i1}, $i = 1,2,3$.

Rule 1

Applying Rule 1, the following (5×5) matrix $(A_{11})_3$ is generated:

More specifically Rule 1.1 generate the first row of $(A_{11})_3$.
Rules 1.2.1, 1.2.2 and 1.2.3 create the second and third row of $(A_{11})_3$.
Rule 1.3 generates the next $2B_K = 2B_3 = 2$ rows of $(A_{11})_3$.

$-\mu_{11}$				
$d_{31}\cdot\mu_{31}$	$-\mu_{11}-\mu_{31}$	$d_{32}\cdot\mu_{31}$		
μ_{32}		$-\mu_{11}-\mu_{32}$		
	$d_{31}\cdot\mu_{31}$		$-\mu_{11}-\mu_{31}$	$d_{32}\cdot\mu_{31}$
	μ_{32}			$-\mu_{11}-\mu_{32}$

Rules 2–5 are contained in a DO-loop which is executed only one time ($T = 3$ to $K = 3$). The counters X, Y and W have the values 1, 2 and 2, respectively.

Rule 2

Rules 2.1 reconstruct sub-matrix $(A_{11})_3$. But now $\mu_{w1} = \mu_{21}$ is subtracted from all the diagonal elements of $(A_{11})_{13}$.
Rule 2.2 creates the square matrix F_Y of order 2 with all the diagonal elements equal to $d_{w1}\mu_{w1} = d_{21}\mu_{21}$.
Rule 2.3 generates the square sub-matrix $(A_{13})_Y = (A_{13})_2$ of order 2.

$-\mu_{11}-\mu_{21}$						
$d_{31}\cdot\mu_{31}$	$-\mu_{11}-\mu_{31}-\mu_{21}$	$d_{32}\cdot\mu_{31}$				
μ_{32}		$-\mu_{11}-\mu_{32}-\mu_{21}$				
	$d_{31}\cdot\mu_{31}$		$-\mu_{11}-\mu_{31}-\mu_{21}$	$d_{32}\cdot\mu_{31}$	$d_{21}\cdot\mu_{21}$	
	μ_{32}			$-\mu_{11}-\mu_{32}-\mu_{21}$		$d_{21}\cdot\mu_{21}$
					$-\mu_{11}-\mu_{31}$	$d_{32}\cdot\mu_{31}$
						$-\mu_{11}-\mu_{32}$

Rule 2.4 generates the diagonal matrix $(A_{16})_2$ of order

$$\frac{M^{Y}_{B_{W-1},\ldots,B_K}}{2} - M^{Y-1}_{B_W,\ldots,B_K} = \frac{M^2_{B_3}}{2} - M^1 = \frac{14}{2} - 2 = 5$$

with all its diagonal elements equal to $d_{w2}\mu_{w1} = d_{22}\mu_{21}$.

$d_{22} \cdot \mu_{21}$				
	$d_{22} \cdot \mu_{21}$			
		$d_{22} \cdot \mu_{21}$		
			$d_{22} \cdot \mu_{21}$	
				$d_{22} \cdot \mu_{21}$

This is the end of the first zone of A_{12} for $K = 3$ stations. Next, the second zone of matrix A_{12} is generated for $K = 3$ stations. Rules 2.1–2.3 are repeated with the change that μ_{w1} is replaced by μ_{w2}.

$-\mu_{11}-\mu_{22}$						
$d_{31} \cdot \mu_{31}$	$-\mu_{11} - \mu_{31} - \mu_{22}$	$d_{32} \cdot \mu_{31}$				
μ_{32}		$-\mu_{11} - \mu_{32} - \mu_{22}$				
	$d_{31} \cdot \mu_{31}$		$-\mu_{11} - \mu_{31} - \mu_{22}$	$d_{32} \cdot \mu_{31}$	$d_{21} \cdot \mu_{21}$	
	μ_{32}			$-\mu_{11} - \mu_{32} - \mu_{22}$		$d_{21} \cdot \mu_{21}$
					$-\mu_{11} - \mu_{31}$	$d_{32} \cdot \mu_{31}$
						$-\mu_{11} - \mu_{32}$

At this stage the $(A_{12})_3$ matrix has been created.

Rule 4

Applying rules 4.1–4.3 the following first zone of sub-matrix $(A_{14})_3$ is generated:

	$d_{21} \cdot \mu_{21}$			
			$d_{21} \cdot \mu_{21}$	
				$d_{21} \cdot \mu_{21}$
			$d_{31} \cdot \mu_{31}$	
			μ_{32}	

The second zone of $(A_{14})_3$ is:

	μ_{22}			
			μ_{22}	
				μ_{22}
			$d_{31}\cdot\mu_{31}$	
			μ_{32}	

At this stage, sub-matrix $(A_{14})_3$ for $K = 3$ stations has been generated.

Rules 2 and 4

$B_W = B_2 = 1$, thus the DO-loop: DO $Z = 1$ to B_W is performed just once, i.e., sub-matrix A_{12} and A_{14} for $K = 3$, are copied just once.

Rule 3

Sub-matrix $(A_{12})_3$ is copied and the element $-\mu_{11}$ is eliminated from the diagonal elements of this matrix, giving the following matrix:

$-\mu_{21}$						
$d_{31}\cdot\mu_{31}$	$-\mu_{31}-\mu_{21}$	$d_{32}\cdot\mu_{31}$				
μ_{32}		$-\mu_{32}-\mu_{21}$				
	$d_{31}\cdot\mu_{31}$		$-\mu_{31}-\mu_{21}$	$d_{32}\cdot\mu_{31}$	$d_{21}\cdot\mu_{21}$	
	μ_{32}			$-\mu_{32}-\mu_{21}$		$d_{21}\cdot\mu_{21}$
					$-\mu_{31}$	$d_{32}\cdot\mu_{31}$
						$-\mu_{32}$

The diagonal sub-matrix of $(A_{13})_3$ is:

$d_{22}\cdot\mu_{21}$				
	$d_{22}\cdot\mu_{21}$			
		$d_{22}\cdot\mu_{21}$		
			$d_{22}\cdot\mu_{21}$	
				$d_{22}\cdot\mu_{21}$

At this stage, the first zone of $(A_{13})_3$ has been created. The second zone of $(A_{13})_3$ is:

$-\mu_{22}$						
$d_{31}\cdot\mu_{31}$	$-\mu_{31}-\mu_{22}$	$d_{32}\cdot\mu_{31}$				
μ_{32}		$-\mu_{32}-\mu_{22}$				
	$d_{31}\cdot\mu_{31}$		$-\mu_{31}-\mu_{22}$	$d_{32}\cdot\mu_{31}$	$d_{21}\cdot\mu_{21}$	
	μ_{32}			$-\mu_{32}-\mu_{22}$		$d_{21}\cdot\mu_{21}$
					$-\mu_{31}$	$d_{32}\cdot\mu_{31}$
						$-\mu_{32}$

Thus, sub-matrix $(A_{13})_3$ has been generated.

Rule 5

Sub-matrix $(A_{15})_3$ is the same as $(A_{14})_3$:

	$d_{21}\cdot\mu_{21}$			
			$d_{21}\cdot\mu_{21}$	
				$d_{21}\cdot\mu_{21}$
			$d_{31}\cdot\mu_{31}$	
			μ_{32}	
	μ_{22}			
			μ_{22}	
				μ_{22}
			$d_{31}\cdot\mu_{31}$	
			μ_{32}	

At this stage, the DO-loop: DO $T = 3$ to $K = 3$ ends.

Rule 6

Sub-matrix $(A_{16})_3$ is a diagonal matrix of order $(M_{1,1}^3/2) - M_1^2 = 94/2 - 14 = 47 - 14 = 33$ with its diagonal elements equal to $d_{12}\cdot\mu_{11}$. At this stage, the first zone of matrix A_1 has been completed. To generate the second zone of A_1, Rules 1, 2, 3, 2 and 4, 3 and 5 are applied again with the following changes:

1. μ_{11} is replaced by μ_{12} and
2. In applying Rule 5 the location of sub-matrix A_{15} is changed. A_{15} is located under A_{15} of the first zone (see Fig. 4).

At this stage, matrix A_1 has been constructed.

Rule 7

This rule generates matrix $(A_2)_3$ which is an almost diagonal matrix and consists of two zones. $(A_2)_3$ has dimensions:

$$\left(M^K_{B_2,\ldots,B_K} - M^{K-1}_{B_3,\ldots,B_K}\right)\frac{M^K_{B_2,\ldots,B_K}}{2} = (94-14)\frac{94}{2} = 80 \times 47.$$

The structure of the first zone of $(A_2)_3$ is shown in Fig. A1.

The shaded matrices are square diagonal matrices of dimensions $m_{1,3} \times m_{1,3} = 14 \times 14$, except the first one, the order of which is 5. All diagonal elements of these sub-matrices are equal to $d_{11} \cdot \mu_{11}$. There is an empty zone of dimension (14×33) between the two zones (see Fig. 8). The second zone has the same structure as the first zone. However, its diagonal elements have the value μ_{12}. The conservative matrix A is given by the sum of the three matrices A_0, A_1 and A_2.

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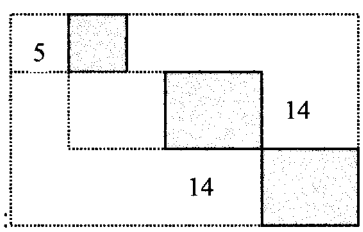

Fig. A1. The structure of A_2 (the first zone).

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