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THE THROUGHPUT OF MULTISTATION PRODUCTION LINES WITH NO INTERMEDIATE BUFFERS

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This paper concerns the throughput rate of multistation reliable production lines with no intermediate buffers. Processing times at the service stations are independent, exponential random variables, possibly with different means. We extend the work started in E. J. Muth (1984) to provide an algorithm that allows for the efficient computation of longer lines and to provide results for the nonidentical server case. A result is presented which provides the distribution function of the holding time at the stations.

Production lines are often modeled as tandem queues with no intermediate buffers. A common method to do this is to model the lines as a series Markov network in the steady state. This requires evaluating the steady state equations over 6,765 states when there are ten stations.

Muth (1977, 1984) developed a recursive method, which yields the throughput as well as the distribution function of the holding, starving, and blocking periods using a Markov process on a smaller state space. We extend this model in this paper.

In Section 1 the holding time model is briefly described by recalling Muth's formulas, and in Section 2, the steps of the proposed approach are summarized. A general formula provides the distribution function of the holding period of the ith station, $i = 2, 3, \dots, K - 1$. In Section 3, numerical results for reliable production lines with up to ten stations in series and exponentially distributed processing times and with no intermediate buffers are tabulated and compared with the Alkaff and Muth results. Section 4 compares our approach with the Alkaff and Muth (1987) method.

1. THE MODEL

Production lines consisting of K single-machine stations linked in series are considered with no intermediate buffers. There is an unlimited supply of items at the first station so that this station can never be empty (idle). Each item enters the line at station 1, passes through all stations in order and leaves the Kth station in finished form. The service times S_i, $i = 1, 2, \dots, K$, are independent exponentially distributed random variables and are not identically distributed. It is assumed that the service stations are reliable and that they can service only one item at a time.

Three periods are defined: the idle period, the busy period, and the blocking period. The *idle period* is the time interval during which a station is idle; the *busy period* is the time interval during which a station is servicing an item and the *blocking period* is the time interval during which a station cannot provide service on an item, because the last item serviced at this station cannot move on to the next station because that station is busy or blocked. It is assumed that the last station (Kth) can never be blocked, i.e., there is a buffer of unlimited capacity out of the last station that can accommodate all the released parts.

The following random variables are defined:

$S_j(n)$: the service period of item n at station j;

$B_j(n)$: the blocking period of item n at station j;

$H_j(n)$: the holding period of item n at station j.

It holds:

$$H_j(n) = S_j(n) + B_j(n); \tag{1}$$

and

$Ij(n)$: the idle period of station j following the departure of item $(n-1)$.

The following recursive relationships among the above random variables have been derived by Muth (1984):

$$H_{j-1}(n) = \max[S_{j-1}(n),\ R_j(n-1)], \tag{2}$$

where $R_j(n)$ is given by

$$R_j(n) = H_j(n) - I_{j-1}(n+1), \quad \text{for } j \in \Lambda \equiv \{1, 2, \dots, K\}. \tag{3}$$

$R_j(n)$ represents the time difference between the departure of item n from station j and the arrival of the next item, $(n + 1)$, at station $(j - 1)$; hence, this random variable can be either positive or negative. It is defined by

$$R_j^+(n) = \max[0,\ R_j(n)], \tag{4}$$

and

$$R_j^-(n) = \min[0,\ R_j(n)], \tag{5}$$

where $R_j^+(n)$ is the residual holding period of item n at station j, following the entrance of item $n + 1$ into station $j - 1$, and the absolute value of $R_j^-(n)$ is the

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time period during which both stations j and $j-1$ are starved together. Since station 1 is never starved, $R_2(n) = H_2(n)$, so $R_2(n)$ is a positive random variable and $R_2^-(n) = 0$.

Another useful relationship is

$$I_j(n) = \max[I_{j-1}(n) + S_{j-1}(n) - H_j(n-1), 0] \quad \text{for } j \in \Lambda. \tag{6}$$

$H_j(n)$, $I_j(n)$ and $S_j(n)$ are defined to be identically zero for $j = 0$ and $j = K + 1$. It also holds that

$$I_1(n) = 0, \tag{7}$$

because the first station can never be idle, and

$$H_K(n) = S_K(n), \tag{8}$$

because the last station can never be blocked. It may be shown that these relationships hold:

$$H_j(n) = \max[S_j(n), R_{j+1}^+(n-1)], \tag{9}$$

and

$$I_j(n) = \max[S_{j-1}(n) - R_j(n-1), 0]. \tag{10}$$

The relationships among the above variables can be conveniently displayed and derived through the use of activity networks, which have proved useful for describing the sample path behavior of any production line (Dallery and Gershwin 1991). These equations are referred to as the *evolution equations* of the production line. They have also proved useful for establishing qualitative properties, such as monotonicity and reversibility of the flow lines.

1.1. The Integral Equations and the Throughput

If all service times have finite mean values, then $R_j(n)$, $I_j(n)$, and $H_j(n)$ converge in distribution to the random variables R_j, I_j and H_j as n tends to infinity. If $H_j(n-1)$ is statistically independent of $I_{j-1}(n)$ (Muth 1984), then from (3) and (4) (see Alkaff and Muth 1987):

$$F_{R_j^+}(t) = \begin{cases} 1 - \int_{x=\max[0,t]=t}^{\infty} F_{I_{j-1}}(x-t)\, dF_{H_j}(x) & \text{if } t \geq 0, \\ 0 & \text{otherwise;} \end{cases} \tag{11}$$

or by changing the variables inside the arguments:

$$F_{R_j^+}(t) = \begin{cases} 1 - \int_{x=\max[0,-t]=0}^{\infty} F_{I_{j-1}}(x)\, dF_{H_j}(x+t) & \text{if } t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

From (3) and (5):

$$F_{R_j^-}(t) = \begin{cases} 1 - \int_{x=0}^{\infty} F_{I_{j-1}}(x-t)\, dF_{H_j}(x) & \text{if } t \leq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{13}$$

If $R_{j+1}^+(n-1)$ is statistically independent of $S_j(n)$, then from (9):

$$F_{H_j}(t) = F_{S_j}(t) F_{R_{j+1}^+}(t), \tag{14}$$

and from (10):

$$F_{I_j}(t) = \begin{cases} 1 - \int_{x=-t}^{\infty} F_{R_j^-}(x)\, dF_{S_{j-1}}(x+t) \\ \quad - \int_{x=0}^{\infty} F_{R_j^+}(x)\, dF_{S_{j-1}}(x+t) & \text{if } t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Substitution of (14) into (12) and (13) gives, for all $j \in \Lambda$:

$$F_{R_j^+}(t) = \begin{cases} 1 - \int_{x=\max[0,-t]=0}^{\infty} F_{I_{j-1}}(x) d(F_{S_j}(x+t) \\ \quad \cdot F_{R_{j+1}^+}(x+t)) & \text{if } t \geq 0, \\ 0 & \text{otherwise;} \end{cases} \tag{16}$$

$$F_{R_j^-}(t) = \begin{cases} 1 - \int_{x=0}^{\infty} F_{I_{j-1}}(x-t) d(F_{S_j}(x) F_{R_{j+1}^+}(x)) & \text{if } t \leq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{17}$$

The throughput or mean production rate of a K-station production line, denoted by X_K is defined as

$$X_K = \lim_{t \to \infty} \frac{E[N(t)]}{t}, \tag{18}$$

where $N(t)$ denotes the number of items released from station K in the time interval $(0, t]$ and $E[*]$ denotes the expected value operator. That is,

$$X_K = \frac{1}{E[H_1]} = \left[\int_{t=0}^{\infty} (1 - F_{H_1}(t))\, dt \right]^{-1}. \tag{19}$$

Equation 2 for $j = 2$ gives:

$$H_1 = \max[S_1(n), R_2^+(n-1)],$$

and from (9) and (4):

$$R_2^+(n-1) = \max[0, R_2(n-1)],$$

where,

$$R_2(n-1) = H_2(n-1) - I_1(n) = H_2(n-1).$$

From (14) it follows that

$$F_{H_1}(t) = F_{S_1}(t) F_{R_2^+}(t). \tag{20}$$

Since the first station is never starved it holds that $R_2^+ = H_2$, and using (14) again:

$$F_{H_1}(t) = F_{S_1}(t) F_{S_2}(t) F_{R_j^+}(t). \tag{21}$$

To compute $F_{H_1}(t)$, $F_{R_3^+}(t)$ has to be determined first from the simultaneous equations (15), (16), and (17), and then the throughput of the line is obtained from (19).

2. THE SOLUTION METHOD

To obtain the throughput of the K-station reliable production line with exponentially distributed processing times, the proposed method involves the following four steps:

STEP 1. Derivation of the integral equations defining the distribution functions of the holding periods of the i stations, $i = 2, 3, \ldots, K-1$.

STEP 2. Transformation of the integral equations to equivalent nonlinear equations.

STEP 3. Solution of the simultaneous nonlinear equations via conventional numerical techniques.

STEP 4. Calculation of the throughput from (19) after having determined $F_{H_1}(t)$ via $F_{R_3^+}(t)$ which in turn is calculated from the simultaneous equations (15), (16), and (17).

Comments

To facilitate the first step, a general formula has been obtained which gives the distribution function of the holding period of the various service stations. For $j = 2, 3, \ldots, K-1$ and $t \geq 0$:

$$
\begin{aligned}
F_{H_j}(t) &= F_j(t)F_{H_{j+1}}(t) \\
&+F_j(t)\int_{x_1=0}^{\infty}\int_{x_2=0}^{\infty} F_{H_j}(x_2)\, dF_{j-1}(x_2+x_1) \\
&\cdot dF_{H_{j+1}}(x_1+t) \\
&+F_j(t)\int_{x_1=0}^{\infty}\int_{x_2=0}^{\infty}\int_{x_3=0}^{\infty}\int_{x_4=0}^{\infty} F_{H_{j-1}}(x_4)dF_{j-2}(x_4+x_3) \\
&\cdot dF_{H_j}(x_3+x_2)\, dF_{j-1}(x_2+x_1)\, dF_{H_{j+1}}(x_1+t) \\
&+ \\
&\vdots \\
&+ \\
&+F_j(t)\int_{x_1=0}^{\infty}\int_{x_2=0}^{\infty}\cdots\int_{x_{\nu-1}=0}^{\infty}\int_{x_\nu=0}^{\infty}{}_{\nu=2(j-1)} \\
&\cdot F_{H_2}(x_\nu)\, dF_1(x_\nu+x_{\nu-1}) \\
&\cdot dF_{H_3}(x_{\nu-1}+x_{\nu-2})dF_2(x_{\nu-2}+x_{\nu-3})\cdots \\
&\cdot dF_{j-1}(x_2+x_1)dF_{H_{j+1}}(x_1+t). \qquad (22)
\end{aligned}
$$

For $j = K$:

$$F_{H_K}(t) = F_{S_K}(t) \equiv F_K(t) = 1 - e^{-\mu_K t}, \qquad (23)$$

and $F_{H_1}(t)$ is given from (21).

There are $j = 2, 3, \ldots, K-1$ terms in the right-hand side of formula (22), $(j-1)$ of which are in the form of integrals. The usefulness of this general formula resides in the fact that by applying it, one can directly derive the distribution functions of the holding periods of stations $2, 3, \ldots, K-1$.

The second step involves tedious algebraic manipulation to transform the integral equations, derived in Step 1, applying formula (22), to equivalent algebraic (nonlinear) equations. At this step, one can identify the drawback of the proposed method against the Alkaff and Muth method, because the mathematical expressions involved are complex, but, on the other hand, they lead to the exact solution which tackles the general case with nonidentical servers and not only the balanced line case.

To facilitate this step, various useful observations have been made. For example, it was found that $F_{R_3^+}(t)$ has the form:

$$
\begin{aligned}
F_{R_3^+}(t) = 1 &- \sum_{\iota_1=1}^{C_1} D_{\iota_1} e^{-s_{\iota_1} t} - \sum_{\iota_2=1}^{C_2} D_{\iota_2} e^{-s_{\iota_2} t} \\
&- \sum_{\iota_3=1}^{C_3} D_{\iota_3} e^{-s_{\iota_3} t} - \cdots - D_{\iota_{K-2}} e^{-s_{\iota_{K-2}} t}, \qquad (24)
\end{aligned}
$$

where

$$C_n \equiv C(K-2, n) = \binom{K-2}{n},$$

and

$$s_{\iota_1} = \mu_\iota, \quad \iota = 3, \ldots, K, \qquad (25)$$

$$s_{\iota_2} = \mu_\iota + \mu_j, \quad j, \iota = 3, \ldots, K \text{ and } j \neq \iota, \qquad (26)$$

$$s_{\iota_3} = \mu_\iota + \mu_j + \mu_\ell, \quad \ell, j, \iota = 3, \ldots, K \text{ and } \ell \neq j \neq \iota, \qquad (27)$$

$$\vdots = \vdots$$

$$s_{\iota_{K-2}} = \sum_{\nu=3}^{K} \mu_\nu, \qquad (28)$$

and

$$\iota_{K-2} = 2^{K-2} - 1. \qquad (29)$$

The number m_K, of the coefficients D_i, is obtained as the sum of all the possible combinations of the $(K-2)$ elements, taken in groups of $1, 2, \ldots, K-2$ (elements). This leads to the following proposition.

Proposition. *The number, m_K, of the coefficients D_i, in the distribution function $F_{R_3^+}(t)$, for a K-station series psroduction line, with exponentially distributed service times and no buffers between successive stations, is given by the formula:*

$$m_K = 2^{K-2} - 1. \qquad (30)$$

In Step 3, we applied the Newton-Raphson method to solve the system of nonlinear equations (Papadopoulos and O'Kelly 1988).

In Step 4, we simply used (19) to derive the throughput of the production line.

3. NUMERICAL RESULTS, COMPARISON WITH THE ALKAFF AND MUTH RESULTS

By applying the steps of the proposed method, given in the previous section, we were able to derive numerical

Table I
Numerical Results

Number of Stations K	Approximate Throughput $(X_K)_{\text{approx.}}$	Exact Throughput $(X_K)_{\text{exact}}$	Absolute Difference D
3	0.5641	0.5641	0.0000
4	0.5126	0.5146	0.0020
5	0.4806	0.4854	0.0048
6	0.4583	0.4662	0.0079
7	0.4415	0.4527	0.0112
8	0.4282	0.4427	0.0145
9	0.4173	0.4350	0.0177
10	0.4081	0.4290	0.0209

results for the throughput of production lines with up to ten service stations. These results match those obtained by Alkaff and Muth (1987) and are given in Table I. For comparison, the case of balanced production lines was treated in all arithmetic examples. Details of the solution of other service time cases are given in Papadopoulos and O'Kelly (1988).

The differences of the approximate results against the exact results are due to the omission of the negative realizations of the random variables $R_j(t)$ and $I_j(t)$, viz., the residual holding periods and starving periods, respectively. The number of equations we had to solve in the approximate case were $K - 2$, while in the exact case these were $(K - 1)(K - 2)/2$. For example, for $K = 10$, the approximate and the exact methods require the solution of only 8 and 36 nonlinear equations, respectively, instead of 6,765 linear equations required in the Markovian state model.

4. CONCLUSION

The holding time model was introduced by Muth (1977, 1984). Compared with the traditional Markovian state model, this method is much more efficient from the computational point of view, because to derive the steady-state probabilities (and from them, the throughput), the number of equations to be solved is reduced greatly. The application of the holding time model method (Muth's method) develops not only the throughput, but also the (cumulative) distribution functions of the holding, starving, and blocking periods. These are derived from the recursive relationships that hold among these random variables.

Alkaff and Muth (1987) provided two methods for deriving the throughput of multistation production lines with stochastic servers, viz., exact and approximate. They presented a technique that used operators, which resulted in the solution of a fixed-point problem.

Our work is a direct extension of Muth (1984). The holding time model is utilized for analyzing multistation reliable production lines with exponentially distributed service times and up to ten service stations. A general formula which provides the cumulative distribution function of the holding period of the service stations of the production line, not previously reported, was presented.

Both procedures (ours and those of Muth et al.) assume that $H_j(n - 1)$ and $I_{j-1}(n)$, viz., the holding and starving periods are statistically independent. The question of whether or not $H_j(n - 1)$ and $I_{j-1}(n)$ are statistically independent when $K > 3$ still remains open. Alkaff and Muth have conducted extensive simulations of 4 and 5 station balanced production lines with exponential service times and their results indicated a slight positive correlation. This topic needs further investigation.

The approach presented here could be applied to the case where the processing times at the various service stations of the production line follow the Erlang distribution or special types of the phase-type distribution, in general. Although this is possible conceptually, in practice this is not feasible for long production lines or even for short production lines with a large number of exponential phases of the service distribution.

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